On the vanishing of negative K-groups

Thomas Geisser · **Lars Hesselholt**

Abstract We show that for a d-dimensional scheme X essentially of finite type over an infinite perfect field k of characteristic $p > 0$, the negative K-groups $K_q(X)$ vanish for $q < -d$ provided that strong resolution of singularities holds over the field k.

Keywords Negative K-groups · topological cyclic homology · cdh-topology

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Introduction

A conjecture of Weibel [22, Question 2.9] predicts that for every noetherian scheme X, the negative K-groups $K_q(X)$ vanish for $q < -\dim(X)$. It was proved recently by Cortiñas, Haesemeyer, Schlichting, and Weibel [4, Theorem 6.2] that the conjecture holds if X is essentially of finite type over a field of characteristic 0. In this paper, we prove similarly that the conjecture holds if X is essentially of finite type over an infinite perfect field k of characteristic $p > 0$ provided that strong resolution of singularities holds over k. The proofs are by comparison with Connes' cyclic homology [18] and the topological cyclic homology of Bökstedt, Hsiang, and Madsen [3], respectively.

We say that strong resolution of singularities holds over k if for every integral scheme X separated and of finite type over k, there exists a sequence of blow-ups

$$X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$$

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such that the reduced scheme X_r^{red} is smooth over k; the center Y_i of the blow-up $X_{i+1} \to X_i$ is connected and smooth over k; the closed embedding of Y_i in X_i is normally flat; and Y_i is nowhere dense in X_i. Strong resolution of singularities holds over fields of characteristic zero by Hironaka [15, Theorem 1*]. In general, the field k must necessarily be perfect. We say that a scheme is essentially of finite type over k if it can be covered by finitely many affine open subsets of the form $\operatorname{Spec} S^{-1}A$ with A a finitely generated k-algebra and $S \subset A$ a multiplicative subset. The following result was conjectured by Weibel [22, Question 2.9]:

Theorem A. *Let k be an infinite perfect field of characteristic $p > 0$ such that strong resolution of singularities holds over k, and let X be a d-dimensional scheme essentially of finite type over k. Then $K_q(X)$ vanishes for $q < -d$.*

In general, the group $K_{-d}(X)$ is non-zero. For instance, by closed Mayer-Vietoris, the group $K_{-d}(\partial\Delta_k^{d+1})$ is readily seen to be an infinite cyclic group. Therefore, the vanishing result above is optimal. We further show in Theorem 5.3 below that, under the assumption that strong resolution of singularities holds over all infinite perfect fields of characteristic $p > 0$, the conclusion of Theorem A is true for any scheme X of finite type over any field of characteristic p.

To prove Theorem A, we consider the cyclotomic trace map

$$\mathrm{tr}\colon K(X) \to \{\mathrm{TC}^n(X;p)\}$$

from the non-connective Bass complete K-theory spectrum of X to the topological cyclic homology pro-spectrum of X and define the pro-spectrum $\{F^n(X)\}$ to be the level-wise mapping fiber; compare [7, Section 1]. Then $\{F^n(-)\}$ defines a presheaf of pro-spectra on the category Sch/X of schemes separated and of finite type over X. We first show that, in the situation of Theorem A, this presheaf satisfies descent with respect to the cdh-topology of Voevodsky [19].

Theorem B. *Let k be an infinite perfect field of positive characteristic p such that strong resolution of singularities holds over k, and let X be a scheme essentially of finite type over k. Then for all integers q, the canonical map*

$$\{F_q^n(X)\} \to \{\mathbb{H}_{\mathrm{cdh}}^{-q}(X, F^n(-))\}$$

is an isomorphism of pro-abelian groups.

We remark that if the dimension of X is zero then Theorem B reduces to the statement that the map $\{F_q^n(X)\} \to \{F_q^n(X^{\mathrm{red}})\}$ induced by the canonical inclusion is an isomorphism of pro-abelian groups. This statement, in turn, is a special case of the general fact that $\{F_q^n(-)\}$ is invariant with respect to nilpotent extensions of unital associative $\mathbb{F}_p$-algebras [7, Theorem B]. It would be very interesting to similarly extend Theorem B to a statement valid for all unital associative $\mathbb{F}_p$-algebras.

To prove Theorem B we generalize of a theorem of Cortiñas, Haesemeyer, Schlichting, and Weibel [4, Theorem 3.12] to show that if strong resolution of singularities holds over the infinite perfect field k, then a presheaf of pro-spectra satisfies cdh-descent for schemes essentially of finite type over k, provided that it takes infinitesimal thinkenings to weak equivalences and finite abstract blow-up squares to homotopy cartesian squares, and provided further that the individual presheaves of spectra

satisfy Nisnevich descent and take squares defined by blow-ups along regular embeddings to homotopy cartesian squares. The presheaf $\{F^n(-)\}$ satisfies all four properties according to theorems of Thomason [20, Theorem 2.1] and [21, Theorem 10.8], Blumberg and Mandell [2, Theorem 1.4], and the authors [7, Theorem B and D]. Hence, Theorem B follows.

In the situation of Theorem A, every cdh-covering of X admits a refinement to a cdh-covering by schemes essentially smooth over k. Together with a cohomological dimension result of Suslin and Voevodsky [19, Theorem 12.5] this shows that the group $\mathbb{H}^{-q}_{\mathrm{cdh}}(X, K(-))$ vanishes for $q < -d$. Therefore, in view of Theorem B, to prove Theorem A, it suffices to prove the following result.

Theorem C. *Let k be a perfect field of positive characteristic p such that resolution of singularities holds over k, and let X be a d-dimensional scheme essentially of finite type over k. Then the canonical map*

$$\{\mathrm{TC}^n_q(X;p)\} \to \{\mathbb{H}^{-q}_{\mathrm{cdh}}(X, \mathrm{TC}^n(-;p))\}$$

is an isomorphism of pro-abelian groups for $q < -d$, and an epimorphism of pro-abelian groups for $q = -d$.

We note that Theorem C uses the weaker assumption that resolution of singularities holds over k: Every integral k-scheme separated and of finite type admits a proper bi-rational morphism $p\colon X' \to X$ from a smooth k-scheme. We expect the map in the statement of Theorem C to be an isomorphism of pro-abelian groups for $q \leqslant -d$, and an epimorphism for $q = -d+1$.

To prove Theorem C, we take advantage of the fact that topological cyclic homology, as opposed to K-theory, satisfies étale descent. This implies that, to prove Theorem C, we may replace the cdh-topology by the finer eh-topology defined in [6, Definition 2.1] to be the smallest Grothendieck topology on Sch/X for which both étale and cdh-coverings are coverings. The proof of Theorem C is then completed by a careful cohomological analysis of the eh-sheaves $a_{\mathrm{eh}}\,\mathrm{TC}^n_q(-;p)$ associated with the presheaves of homotopy groups $\mathrm{TC}^n_q(-;p) = \pi_q\,\mathrm{TC}^n(-;p)$ in combination with the following cohomological dimension result.

Theorem D. *Let X be a scheme essentially of finite type over a field k of characteristic $p > 0$. Then the p-cohomological dimension of X with respect to the* eh*-topology is less than or equal to* $\dim(X)+1$.

We remark that Theorems B and C both concern pre-sheaves of pro-spectra. Indeed, we do not know whether the analog of Theorem C holds if the pre-sheaf of pro-spectra $\{\mathrm{TC}^n(-;p)\}$ is replaced by the pre-sheaf of spectra $\mathrm{TC}(-;p)$ given by the homotopy limit. Our recent paper [7] was written primarily with the purpose of proving Theorem B above.

Let X be a noetherian scheme. We define Sch/X to be the category of schemes separated and of finite type over X and denote by $a_\tau\colon (\mathrm{Sch}/X)^\wedge \to (\mathrm{Sch}/X)^\sim_\tau$ and $i_\tau\colon (\mathrm{Sch}/X)^\sim_\tau \to (\mathrm{Sch}/X)^\wedge$ the sheafification functor and the inclusion functor, respectively, between the categories of presheaves and τ-sheaves of sets. We further denote by $\alpha\colon (\mathrm{Sch}/X)_{\mathrm{eh}} \to (\mathrm{Sch}/X)_{\mathrm{et}}$ the canonical morphism of sites.

1 cdh-descent

In this section, we formulate and prove a generalization of [4, Theorem 3.12] to pre-sheaves of pro-spectra. We apply this theorem to prove Theorem B of the introduction. We first recall some definitions.

Let X be a noetherian scheme, and let $F(-)$ be a presheaf of fibrant symmetric spectra on Sch/X. If τ is a Grothendieck topology on Sch/X that has enough points, we define the hypercohomology spectrum $\mathbb{H}^{\cdot}_{\tau}(X,F(-))$ to be the Godement-Thomason construction [20, Definition 1.33] of the site $(\mathrm{Sch}/X)_{\tau}$ with coefficients in the presheaf $F(-)$; see also [8, Section 3.1]. We recall from [8, Proposition 3.1.2] that, in this situation, there is a conditionally convergent spectral sequence

$$E^2_{s,t} = H^{-s}_{\tau}(X, a_{\tau}F_t(-)) \Rightarrow \mathbb{H}^{-s-t}_{\tau}(X,F(-))$$

from the sheaf cohomology groups of X with coefficients in the τ-sheaf on Sch/X associated with the presheaf $F_t(-) = \pi_t(F(-))$ and with abutment the homotopy groups $\mathbb{H}^{-q}_{\tau}(X,F(-)) = \pi_q\mathbb{H}^{\cdot}_{\tau}(X,F(-))$.

We next recall the cdh-topology on Sch/X from [19]. We say that the cartesian square of X-schemes

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "p'"] & Y' \arrow[d, "p"] \\ Z \arrow[r, "i"] & Y \end{tikzcd}$$

is an *abstract blow-up square* if i is a closed immersion and p is a proper map that induces an isomorphism of $Y' \smallsetminus Z'$ onto $Y \smallsetminus Z$. We say that the square is a *finite abstract blow-up square* if it is an abstract blow-up square and the proper map p is finite. We say that the square is an elementary Nisnevich square if i is an open immersion and p is an étale map that induces an isomorphism of $Y' \smallsetminus Z'$ onto $Y \smallsetminus Z$. We say that $i\colon Z \to Y$ is an *infinitesimal thickening* if i is a closed immersion and the corresponding quasi-coherent ideal $\mathscr{I} \subset \mathscr{O}_Y$ is nilpotent. The cdh-topology on Sch/X is defined to be the smallest Grothendieck topology such that for every abstract blow-up square and every elementary Nisnevich square, the family of morphisms

$$\{p\colon Y' \to Y, i\colon Z \to Y\}$$

is a covering of Y. In particular, the closed covering of the scheme Y by its irreducible components is a cdh-covering as is the closed immersion $Y^{\mathrm{red}} \to Y$.

For the purpose of this paper, we define a *pro-spectrum* to be a functor from the partially ordered set of positive integers viewed as a category with a single morphism from m to n, if $n \leqslant m$, to the category of fibrant symmetric spectra, and we define a strict map of pro-spectra to be a natural transformation. We define the strict map of pro-spectra $f\colon \{X^n\} \to \{Y^n\}$ to be a *weak equivalence* if for every integer q, the induced map of homotopy groups

$$f_*\colon \{\pi_q(X^n)\} \to \{\pi_q(Y^n)\}$$

is an isomorphism of pro-abelian groups. This, we recall, means that for every n, there exists $m \geqslant n$, such that the maps induced by the structure maps from the kernel and

cokernel of the map $f^m_*\colon \pi_q(X^m) \to \pi_q(Y^m)$ to the kernel and cokernel, respectively, of the map $f^n_*\colon \pi_q(X^n) \to \pi_q(Y^n)$ are both zero. We say that the square diagram of strict maps of pro-spectra

$$\begin{array}{ccc} \{X^n\} & \longrightarrow & \{Y^n\} \\ \downarrow & & \downarrow \\ \{Z^n\} & \longrightarrow & \{W^n\} \end{array}$$

is *homotopy cartesian* if the canonical map

$$\{X^n\} \to \{\operatorname{holim}(Y^n \to W^n \leftarrow Z^n)\}$$

is a weak equivalence.

The following result generalizes [11, Theorem 6.4] and [4, Theorem 3.12].

Theorem 1.1. *Let k be an infinite perfect field such that strong resolution of singularities holds over k, and let $\{F^n(-)\}$ be a presheaf of pro-spectra on the category of schemes essentially of finite type over k. Assume that $\{F^n(-)\}$ takes infinitesimal thickenings to weak equivalences and finite abstract blow-up squares to homotopy cartesian squares. Assume further that each $F^n(-)$ takes elementary Nisnevich squares and squares associated with blow-ups along regular embeddings to homotopy cartesian squares. Then the canonical map defines a weak equivalence*

$$\{F^n(X)\} \xrightarrow{\sim} \{\mathbb{H}^{\cdot}_{\mathrm{cdh}}(X, F^n(-))\}$$

of pro-spectra for every scheme X essentially of finite type over k.

Proof. The proof in outline is analogous to the proof of [11, Theorem 6.4]. But some extra care is needed, since for a map $\{A^n(-)\} \to \{B^n(-)\}$ of pro-objects in the category of sheaves of abelian groups on the category of schemes essentially of finite type over k to be an isomorphism, it does not suffice to show that for every such scheme X and every point $x \in X$, the map $\{A^n(-)_{X,x}\} \to \{B^n(-)_{X,x}\}$ of the pro-abelian groups of stalks is an isomorphism. Instead, one must show that for every scheme X essentially of finite type over k and every point $x \in X$, there exists a Zariski open neighborhood $x \in U \subset X$ such that the map $\{A^n(U)\} \to \{B^n(U)\}$ is an isomorphism of pro-abelian groups. We point out the necessary changes in the proof of loc. cit.

First, it follows from [4, Corollary 3.9] that for every scheme X smooth over k and positive integer n, the canonical map defines a weak equivalence of spectra

$$F^n(X) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{\mathrm{cdh}}(X, F^n(-)).$$

In particular, the canonical map defines a weak equivalence of pro-spectra

$$\{F^n(X)\} \xrightarrow{\sim} \{\mathbb{H}^{\cdot}_{\mathrm{cdh}}(X, F^n(-))\}.$$

Now, it follows verbatim from the proof of [11, Proposition 3.12] that the statement holds for every scheme X which is a normal crossing scheme over k in the sense of loc. cit., Definition 3.10.

Next, let X be a Cohen-Macaulay scheme over k and let $D \subset X$ be an integral subscheme along which X is normally flat. Let X' be the blow-up of X along D and

let D' be the exceptional fiber. We indicate the changes necessary to the proof of op. cit., Theorem 5.7, in order to show that the square of pro-spectra

$$\begin{tikzcd}
\{F^n(D')\} & \{F^n(X')\} \arrow[l] \\
\{F^n(D)\} \arrow[u] & \{F^n(X)\} \arrow[l] \arrow[u]
\end{tikzcd}$$

is homotopy cartesian. It follows from loc. cit., Proposition 5.4, that after replacing X by a Zariski open neighborhood of a given point $x \in X$, there exists a reduction $\tilde{D}$ of D in the sense of loc. cit., Definition 5.1, such that $\tilde{D}$ is regularly embedded in X. Let $X_{\tilde{D}}$ be the blow-up of X along $\tilde{D}$ and let $\tilde{D}'$ be the exceptional fiber. We consider the following diagram of schemes, where every square is cartesian, and the induced diagram of pro-spectra.

$$\begin{tikzcd}
D' \arrow[r, hook] \arrow[dd] & \tilde{D}'' \arrow[r, hook] \arrow[d] & X' \arrow[d] \\
& \tilde{D}' \arrow[r, hook] \arrow[d] & X_{\tilde{D}} \arrow[d] \\
D \arrow[r, hook] & \tilde{D} \arrow[r, hook] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\{F^n(D')\} & \{F^n(\tilde{D}'')\} \arrow[l] & \{F^n(X')\} \arrow[l] \\
& \{F^n(\tilde{D}')\} \arrow[u] & \{F^n(X_{\tilde{D}})\} \arrow[l] \arrow[u] \\
\{F^n(D)\} \arrow[uu] & \{F^n(\tilde{D})\} \arrow[l] \arrow[u] & \{F^n(X)\} \arrow[l] \arrow[u]
\end{tikzcd}$$

The lower right-hand square in the left-hand diagram is a blow-up along a regular embedding, and therefore, the lower right-hand square in the right-hand diagram is homotopy cartesian by assumption. Similarly, the upper right-hand square in the left-hand diagram is a finite abstract blow-up square, and therefore, the upper right-hand square in the right-hand diagram also is homotopy cartesian. Moreover, the left-hand horizontal maps in the left-hand diagram are infinitesimal thickenings, and therefore, the left-hand horizontal maps in the right-hand diagram are weak equivalences. It follows that the outer square in the right-hand diagram is homotopy cartesian as desired.

Now, it follows verbatim from the proof of op. cit., Theorem 6.1, that the theorem holds, if X is a hypersurface in a scheme essentially smooth over k. Similarly, the proof of op. cit., Corollary 6.2, shows that the theorem holds, if X is a local complete intersection in a scheme essentially smooth over k.

Finally, in the general case, we proceed as in the proof of op. cit., Theorem 6.4. We may assume that X is integral, since the presheaf $\{F^n(-)\}$ is invariant under infinitesimal thickenings and satisfies descent for finite closed coverings. We now argue by induction on the dimension d of X. The case $d = 0$ follows from what was proved earlier since X is smooth over k, k being perfect. So we let $d > 0$ and assume the theorem has been proved for schemes of smaller dimension. Replacing X by an affine open neighborhood of a given point $x \in X$, we may embed X as a closed subscheme $X = Z(\mathfrak{p}) \subset U$ of an affine scheme $U = \operatorname{Spec} A$ essentially smooth over k. Since U is regular, there exists a regular sequence in $\mathfrak{p}$ of length $\operatorname{ht}(\mathfrak{p})$. This sequence defines a closed subscheme $\tilde{X} \subset U$ which contains X as an irreducible component

and which, after possibly replacing U by a smaller open neighborhood of x, is a local complete intersection. Hence, the theorem holds for $\tilde{X}$ by what was proved above. Let $X^c \subset \tilde{X}$ be the union of the components other than X. Then the theorem also holds for the intersection $X \cap X^c$ by induction. Therefore, it holds for X (and for X^c) by the Mayer-Vietoris sequence associated with the closed covering of $\tilde{X}$ by X and X^c. $\square$

Remark 1.2. In the situation of Theorem B, suppose that $\{F^n(-)\}$ takes elementary Nisnevich squares of schemes essentially of finite type over k to homotopy cartesian squares. Then for every scheme X essentially of finite type over k, the canonical map defines a weak equivalence

$$\{F^n(X)\} \xrightarrow{\sim} \{\mathbb{H}^{\cdot}_{\mathrm{Nis}}(X, F^n(-))\}.$$

Therefore, if $\{F^n(-)\}$ takes infinitesimal thickenings (resp. finite blow-up squares) of affine schemes essentially of finite type over k to weak equivalences (resp. homotopy cartesian squares) then $\{F^n(-)\}$ takes infinitesimal thickenings (resp. finite blow-up squares) of all schemes essentially of finite type over k to weak equivalences (resp. homotopy cartesian squares).

Proof of Theorem B. We show that the presheaf of pro-spectra $\{F^n(-)\}$, where $F^n(X)$ is the mapping fiber of the cyclotomic trace map

$$\mathrm{tr}\colon K(X) \to \mathrm{TC}^n(X;p),$$

satisfies the hypothesis of Theorem 1.1. The functors $K(-)$ and $\mathrm{TC}^n(-;p)$ both take elementary Nisnevich squares to homotopy cartesian squares. Indeed, this is proved for $K(-)$ in [21, Theorem 10.8] and for $\mathrm{TC}^n(-;p)$ in [8, Proposition 3.2.1]. Therefore, the functor $F^n(-)$ takes elementary Nisnevich squares to homotopy cartesian squares. Next, it follows from Remark 1.2 and from [7, Theorem B] that $\{F^n(-)\}$ takes infinitesimal thickenings to weak equivalences. Similarly, Remarks 1.2 and [7, Theorem D] show that $\{F^n(-)\}$ takes finite abstract blow-up squares to homotopy cartesian squares. Finally, two theorems of Thomason [20, Theorem 2.1] and Blumberg and Mandell [2, Theorem 1.4] show that the functors $K(-)$ and $\mathrm{TC}^n(-,;p)$ take squares associated with blow-ups along regular embeddings to homotopy cartesian squares. Hence, the same holds for the functor $F^n(-)$. Now, Theorem B follows from Theorem 1.1. $\square$

2 The eh-topology

Let X be a noetherian scheme. We recall from [6, Definition 2.1] that the eh-topology on the category Sch/X of schemes separated and of finite type over X is defined to be the smallest Grothendieck topology for which both étale coverings and cdh-coverings are coverings. In this section, we estimate the cohomological dimension of the eh-topology. Our arguments closely follow Suslin-Voevodsky [19, Section 12].

Lemma 2.1. *Let X be a noetherian scheme, and let F be an* eh-*sheaf of abelian groups on* Sch/X. *Then the canonical closed immersion induces an isomorphism*

$$H^*_{\mathrm{eh}}(X,F) \xrightarrow{\sim} H^*_{\mathrm{eh}}(X^{\mathrm{red}},F).$$

Proof. We choose an injective resolution $F \to I^{\cdot}$ in the category of eh-sheaves of abelian groups on Sch/X. Since the closed immersion $i\colon X^{\mathrm{red}} \to X$ is a cdh-covering, we have the equalizer diagram

$$I^{\cdot}(X) \xrightarrow{i^*} I^{\cdot}(X^{\mathrm{red}}) \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} I^{\cdot}(X^{\mathrm{red}} \times_X X^{\mathrm{red}}).$$

But the closed immersion $i\colon X^{\mathrm{red}} \to X$ is also a universal homeomorphism so the two projection pr_1 and pr_2 are equal. Hence, the map i^* is an isomorphism of complexes of abelian groups. The lemma follows. $\square$

For every morphism $f\colon Y \to X$ between noetherian schemes, we have the induced functor $f^{-1}\colon \mathrm{Sch}/X \to \mathrm{Sch}/Y$ defined by $f^{-1}(X'/X) = X' \times_X Y/Y$. This functor, in turn, gives rise to the adjoint pair of functors

$$(\mathrm{Sch}/X)^{\wedge} \underset{f_p}{\overset{f^p}{\rightleftarrows}} (\mathrm{Sch}/Y)^{\wedge}$$

where f_p is the restriction along f^{-1} and f^p the left Kan extension along f^{-1}. Since the functor f_p preserves eh-sheaves, we obtain a morphism of sites

$$f\colon (\mathrm{Sch}/Y)_{\mathrm{eh}} \to (\mathrm{Sch}/X)_{\mathrm{eh}}$$

with the direct image functor f_* given by the restriction of the functor f_p and with the inverse image functor given by $f^* = a_{\mathrm{eh}} f^p i_{\mathrm{eh}}$.

We consider two special cases. First, if $f\colon Y \to X$ is separated and of finite type, the inverse image functor is given by $f^*F(Y'/Y) = F(Y'/X)$ and has an exact left adjoint functor $f_!$. Hence, in this case, we see that f^* preserves injectives and that the canonical map defines an isomorphism

$$H^*_{\mathrm{eh}}(Y,F) \xrightarrow{\sim} H^*_{\mathrm{eh}}(Y, f^*F).$$

Second, suppose that $f\colon Y \to X$ is the limit of a cofiltered diagram $\{X_i\}$ with affine transition maps of schemes separated and of finite type over X. It then follows from [10, Theorem IV.8.8.2] that for every scheme Y' separated and of finite type over Y, there exists an index i and a scheme X_i' separated and of finite type over X_i such that $Y' = X_i' \times_{X_i} Y$. Moreover, in this situation, loc. cit. implies that

$$f^pF(Y'/Y) = \operatorname*{colim}_{j \to i} F(X_i' \times_{X_i} X_j/X).$$

It is proved in [19, Section 12] that, in this case, the functor f^p preserves eh-sheaves, and hence, that the inverse image functor f^* is given by the same formula.

Proposition 2.2. *Let X be a noetherian scheme, and let $f\colon Y \to X$ be the limit of a cofiltered diagram $\{X_i\}$ with affine transition maps of schemes separated and of finite type over X. Then for every* eh*-sheaf F of abelian groups on* Sch/X*, the canonical map defines an isomorphism*

$$\operatorname*{colim}_i H^*_{\mathrm{eh}}(X_i, F) \xrightarrow{\sim} H^*_{\mathrm{eh}}(Y, f^*F).$$

Proof. Let $F \to I^{\cdot}$ be an injective resolution in the category of eh-sheaves of abelian groups on Sch/X. Then, since filtered colimits and finite limits of diagrams of sets commute, the explicit formula for the functor f^* gives an isomorphism

$$\operatorname*{colim}_i H^*_{\mathrm{eh}}(X_i, F) = \operatorname*{colim}_i H^*\Gamma(X_i, I^{\cdot}) \xrightarrow{\sim} H^*(\operatorname*{colim}_i \Gamma(X_i, I^{\cdot})) = H^*\Gamma(Y, f^*I^{\cdot}).$$

We claim that for every injective eh-sheaf I of abelian groups on Sch/X, the eh-sheaf f^*I of abelian groups on Sch/Y is $\Gamma(Y,-)$-acyclic. To show this, it suffices to show that the Čech cohomology groups of $\check{H}^*(Y, f^*I)$ vanish [1, Proposition V.4.3]. Since Y is noetherian, every eh-covering of Y admits a refinement by a finite eh-covering. Therefore, it suffices to show that the Čech cohomology groups of f^*I with respect to every finite eh-covering of Y vanish. But this follows immediately from the explicit formula for the functor f^*, since filtered colimits and finite limits of diagrams of sets commute. This proves the claim. Since f^* is exact, we conclude that $f^*F \to f^*I^{\cdot}$ is a resolution by $\Gamma(Y,-)$-acyclic objects in the category of eh-sheaves of abelian groups on Sch/Y. Therefore, the canonical map

$$H^*_{\mathrm{eh}}(Y, f^*F) \to H^*\Gamma(Y, f^*I^{\cdot})$$

is an isomorphism. This shows that the map of the statement is an isomorphism. □

Theorem 2.3. *Let X be a scheme separated and of finite type over a separably closed field k of positive characteristic p. Then the p-cohomological dimension of X with respect to the* eh-*topology is less than or equal to* $\dim(X)$.

Proof. We must show that for every eh-sheaf F of p-primary torsion abelian groups on Sch/X, the cohomology group $H^q_{\mathrm{eh}}(X,F)$ vanishes for $q > d = \dim(X)$. We follow the proof of [19, Theorem 12.5] and proceed by induction on d. Suppose first that $d = 0$. By Lemma 2.1 we may further assume that X is reduced. It follows from [6, Proposition 2.3] that every eh-covering of X admits a refinement by an étale covering of X. Therefore, we conclude from [1, Theorem III.4.1] that the change-of-topology map defines an isomorphism

$$H^q_{\mathrm{et}}(X, \alpha_*F) \xrightarrow{\sim} H^q_{\mathrm{eh}}(X, F).$$

The statement now follows from [1, X Corollary 5.2].

We next let $d > 0$ and assume that the statement has been proved for schemes of smaller dimension. By Lemma 2.1 we may assume that X is reduced, and by a descending induction on the number of connected components we may further assume that X is integral [6, Prop 2.3]. We consider the Leray spectral sequence

$$E_2^{s,t} = H^s_{\mathrm{et}}(X, R^t\alpha_*F) \Rightarrow H^{s+t}_{\mathrm{eh}}(X, F).$$

Let $x \in X$ be a point of codimension $c < d$, let $\bar{x}$ be a geometric point lying above x, and let $f\colon \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\bar{x}} \to X$ be the canonical map. It follows from Proposition 2.2 that the stalk of $R^t\alpha_*F$ at $\bar{x}$ is given by

$$(R^t\alpha_*F)_{\bar{x}} = H^t_{\mathrm{eh}}(\operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\bar{x}}, f^*F).$$

By [1, X Lemma 3.3 (i)] there exists a morphism $g\colon X' \to X$ from a c-dimensional scheme separated and of finite type over a separably closed field k' and a point $x' \in X'$ lying under $\bar{x}$ such that $g(x') = x$ and such that the induced map

$$g'\colon \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X',\bar{x}} \to \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\bar{x}}$$

is an isomorphism. Let $f'\colon \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X',\bar{x}} \to X'$ be the canonical map. Then

$$(R^t \alpha_* F)_{\bar{x}} = H^t_{\mathrm{eh}}(\operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X',\bar{x}}, g'^* f^* F) = H^t_{\mathrm{eh}}(\operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X',\bar{x}}, f'^* g^* F).$$

Hence, we conclude from Proposition 2.2 that

$$(R^t \alpha_* F)_{\bar{x}} = \operatorname{colim} H^t_{\mathrm{eh}}(U', g^* F)$$

where the colimit ranges over all étale neighborhoods $U' \to X'$ of $\bar{x}$. Each U' is a c-dimensional scheme separated and of finite type over the separably closed field k'. Therefore, the inductive hypothesis shows that $(R^t \alpha_* F)_{\bar{x}}$ is zero for $t > c$. It follows that the sheaf $R^t \alpha_* F$ is supported in dimension $\max\{0, d-t\}$. We recall from [1, X Corollary 5.2] that if Z is a scheme of finite type over a separably closed field of characteristic $p > 0$, then the étale p-cohomological dimension of Z is less than or equal to $\dim(Z)$. This shows that in the Leray spectral sequence, $E_2^{s,t}$ is zero for $s > \max\{0, d-t\}$, and hence, the edge-homomorphism

$$H^q_{\mathrm{eh}}(X, F) \to H^0_{\mathrm{et}}(X, R^q \alpha_* F)$$

is an isomorphism for $q > d$.

We now fix a cohomology class $h \in H^q_{\mathrm{eh}}(X, F)$ with $q > d$ and proceed to show that h is zero. There exists an eh-covering $Y \to X$ such that the restriction of h to Y is zero. Moreover, by [6, Proposition 2.3], the covering $Y \to X$ admits a refinement of the form $U' \to X' \to X$, where $U' \to X'$ is an étale covering and $X' \to X$ a proper bi-rational cdh-covering. We let $X'' \subset X'$ be the closure of the inverse image of the generic point of X, and let $U'' = U \times_{X'} X'' \to X''$. It follows from [19, Lemma 12.4] that X'' is a scheme separated and of finite type over k of dimension at most d and that the morphism $X'' \to X$ is proper and bi-rational. We claim that the restriction h'' of the class h to X'' is zero. Indeed, the restriction of h'' to U'' is zero, and therefore, the image of h'' by the edge homomorphism of the Leray spectral sequence

$$H^q_{\mathrm{eh}}(X'', F) \to H^0_{\mathrm{et}}(X'', R^q \alpha_* F)$$

is zero. But we proved above that the edge homomorphism is an isomorphism, so we find that h'' is zero as claimed. To conclude that h is zero, we choose a proper closed subscheme $Z \subset X$ such that the morphism $X'' \to X$ is an isomorphism outside Z and define $Z'' = X'' \times_X Z$. Then by [6, Proposition 3.2], we have a long exact cohomology sequence

$$\cdots \to H^{q-1}_{\mathrm{eh}}(Z'', F) \to H^q_{\mathrm{eh}}(X, F) \to H^q_{\mathrm{eh}}(Z, F) \oplus H^q_{\mathrm{eh}}(X'', F) \to \cdots$$

The schemes Z and Z'' are of finite type over k and their dimensions are strictly smaller than d. Therefore, by the inductive hypothesis, the restriction map

$$H^q_{\mathrm{eh}}(X, F) \to H^q_{\mathrm{eh}}(X'', F)$$

is an isomorphism for $q > d$. Since the image h'' of h by this map is zero, we conclude that h is zero as stated. This completes the proof. $\square$

Proof of Theorem D. We consider the Leray spectral sequence

$$E_2^{s,t} = H^s_{\mathrm{et}}(X, R^t\alpha_*F) \Rightarrow H^{s+t}_{\mathrm{eh}}(X,F).$$

Let $x \in X$ be a point of codimension $c \leqslant d$, and let $\bar{x}$ be a geometric point lying above x. We claim that stalk of $R^t\alpha_*F$ at $\bar{x}$ vanishes for $t > c$. To prove the claim, we may assume that X is affine. We write X as a localization $j\colon X \to X_1$ of a scheme X_1 separated and of finite type over k. Then $x_1 = j(x) \subset X_1$ is a point of codimension c. Hence, we find as in the proof of Theorem 2.3, that the stalk may be rewritten as a filtered colimit

$$(R^t\alpha_*F)_{\bar{x}} = \operatorname{colim} H^t_{\mathrm{eh}}(U', g^*F)$$

where the U' are c-dimensional schemes separated and of finite type over a separably closed field k'. The claim now follows from Theorem 2.3. We conclude that the sheaf $R^t\alpha_*F$ is zero for $t > d$, and is supported in dimension $d-t$ for $t \leqslant d$. Finally, we recall from [1, X Theorem 5.1] that the étale p-cohomological dimension of a noetherian $\mathbb{F}_p$-scheme Z is less than or equal to $\dim(Z)+1$. Therefore, $E_2^{s,t}$ is zero for $s+t > d+1$. This completes the proof. $\square$

3 The de Rham-Witt sheaves

We say that the scheme X is essentially smooth over the field k if it can be covered by finitely many affine open subsets of the form $\operatorname{Spec} S^{-1}A$ with A a smooth k-algebra and $S \subset A$ a multiplicative set. If X is a scheme essentially smooth over a field k, we let Sm/X be the full subcategory of Sch/X whose objects are the schemes smooth over X. We define the Zariski, étale, cdh, and eh-topology on Sm/X to be the Grothendieck topology induced by the Zariski, étale, cdh, and eh-topology on Sch/X, respectively, in the sense of [1, III.3.1]. We denote by ϕ_p the restriction functor from the category of presheaves on Sch/X to the category of presheaves on Sm/X.

Lemma 3.1. *Let k be a perfect field of positive characteristic p such that resolution of singularities holds over k, and let X be a scheme essentially smooth over k. Then for every presheaf F of abelian groups on* Sch/X*, the canonical map*

$$H^*_\tau(X, a_\tau\phi_pF) \to H^*_\tau(X, a_\tau F)$$

is an isomorphism for τ the Zariski, étale, cdh, *and* eh-*topology.*

Proof. Let ϕ^l and ϕ^r be the left and right adjoint functors of ϕ_p given by the two Kan extensions. The functor ϕ_p preserves τ-sheaves by the definition of the induced Grothendieck topology [1, III.3.1]. We claim that also ϕ^r preserves τ-sheaves. Indeed, for the Zariski topology and étale topology this follows from [1, Corollary III.3.3],

and for the cdh-topology and eh-topology it follows from [1, Theorem III.4.1]. Hence, we get the diagrams of adjunctions

$$
\begin{tikzcd}
(\mathrm{Sch}/X)^{\wedge} \arrow[r, shift left, "a_\tau"] \arrow[d, shift left, "\phi_p"] & (\mathrm{Sch}/X)^{\sim}_{\tau} \arrow[l, shift left, "i_\tau"] \arrow[d, shift left, "\phi_*"] \\
(\mathrm{Sm}/X)^{\wedge} \arrow[r, shift left, "a_\tau"] \arrow[u, shift left, "\phi^l"] & (\mathrm{Sm}/X)^{\sim}_{\tau} \arrow[l, shift left, "i_\tau"] \arrow[u, shift left, "\phi^*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{Sm}/X)^{\wedge} \arrow[r, shift left, "a_\tau"] \arrow[d, shift left, "\phi^r"] & (\mathrm{Sm}/X)^{\sim}_{\tau} \arrow[l, shift left, "i_\tau"] \arrow[d, shift left, "\phi^!"] \\
(\mathrm{Sch}/X)^{\wedge} \arrow[r, shift left, "a_\tau"] \arrow[u, shift left, "\phi_p"] & (\mathrm{Sch}/X)^{\sim}_{\tau} \arrow[l, shift left, "i_\tau"] \arrow[u, shift left, "\phi_*"]
\end{tikzcd}
$$

where ϕ_* and $\phi^!$ are the restrictions of ϕ_p and ϕ^r, respectively, to the subcategory of τ-sheaves, and where $\phi^* = a_\tau \phi^l i_\tau$. It follows that the functor ϕ_* preserves all limits and colimits. Moreover, since the two diagrams of right adjoint functors commute, the two diagrams of left adjoint functors commute up to natural isomorphism. In particular, we have a natural isomorphism

$$a_\tau \phi_p F \xrightarrow{\sim} \phi_* a_\tau F.$$

Now, the map of the statement is equal to the composition

$$H^*_\tau(X, a_\tau \phi_p F) \to H^*_\tau(X, \phi_* a_\tau F) \to H^*_\tau(X, a_\tau F)$$

of the induced isomorphism of cohomology groups and the edge-homomorphism of the Leray spectral sequence

$$E_2^{s,t} = H^s_\tau(X, R^t\phi_* a_\tau F) \Rightarrow H^{s+t}_\tau(X, a_\tau F).$$

Since ϕ_* is exact the spectral sequence collapses and the edge-homomorphism is an isomorphism. This completes the proof. $\square$

We let X be a noetherian $\mathbb{F}_p$-scheme and let $\mathscr{O}_X$ be the presheaf on Sch/X that to the X-scheme X' assigns the $\mathbb{F}_p$-algebra $\Gamma(X', \mathscr{O}_{X'})$. It is a sheaf for the étale topology, but not for the cdh-topology. We recall the presheaf

$$W_n\Omega^q_X = W_n\Omega^q_{\mathscr{O}_X}$$

of de Rham-Witt forms of Bloch-Deligne-Illusie [16, Definition I.1.4]. It follows from Proposition I.1.14 of op. cit. that the associated sheaves $a_{\mathrm{Zar}} W_n\Omega^q_X$ and $a_{\mathrm{et}} W_n\Omega^q_X$ agree and are quasi-coherent sheaves of $W_n(\mathscr{O}_X)$-modules on Sch/X.

Now, suppose that X is a scheme essentially smooth over a perfect field k of characteristic $p > 0$. We recall the structure of the sheaf $a_{\mathrm{et}} W_n\Omega^q_X$ from [16]. We will abuse notation and write $W_n\Omega^q_X$ for the étale sheaf $a_{\mathrm{et}} W_n\Omega^q_X$ on Sch/X. There is a short exact sequence of sheaves of abelian groups on Sm/X for the étale topology

$$0 \to \mathrm{gr}^{n-1} W_n\Omega^q_X \to W_n\Omega^q_X \xrightarrow{R} W_{n-1}\Omega^q_X \to 0 \tag{3.2}$$

where $\mathrm{gr}^{n-1} W_n\Omega^q_X$ is the subsheaf generated by the images of $V^{n-1}\colon \Omega^q_X \to W_n\Omega^q_X$ and $dV^{n-1}\colon \Omega^{q-1}_X \to W_n\Omega^q_X$. Let $Z\Omega^q_X$ and $B\Omega^{q+1}_X$ be the kernel and image sheaves of the differential $d\colon \Omega^q_X \to \Omega^{q+1}_X$. The inverse Cartier operator

$$C^{-1}\colon \Omega^q_X \to Z\Omega^q_X / B\Omega^q_X$$

is defined as follows. Let $F\colon W_2\Omega^q_X \to \Omega^q_X$ be the Frobenius map. It satisfies the relations $dF = pFd$, $FV = p$, and $FdV = d$. The first relation shows that F factors through the inclusion of the subsheaf $Z\Omega^q_X$ in Ω^q_X, and the two remaining relations show that the composition $W_2\Omega^q_X \to Z\Omega^q_X \to Z\Omega^q_X/B\Omega^q_X$ of the Frobenius map and the canonical projection annihilates the subsheaf $\mathrm{gr}^1 W_2\Omega^q_X$. The inverse Cartier operator is now defined to be the composition

$$\Omega^q_X \xleftarrow{\sim} W_2\Omega^q_X/\,\mathrm{gr}^1 W_2\Omega^q_X \xrightarrow{\bar{F}} Z\Omega^q_X/B\Omega^q_X.$$

It is an isomorphism of sheaves of abelian groups on Sm/X for the étale topology; see [17, Theorem 7.1]. The inverse isomorphism

$$C\colon Z\Omega^q_X/B\Omega^q_X \xrightarrow{\sim} \Omega^q_X$$

is the Cartier operator. It gives rise to a chain of subsheaves of abelian groups

$$0 = B_0\Omega^q_X \subset B_1\Omega^1_X \subset \cdots \subset B_s\Omega^q_X \subset \cdots \subset Z_s\Omega^q_X \subset \cdots Z_1\Omega^q_X \subset Z_0\Omega^q_X = \Omega^q_X$$

where $B_0\Omega^q_X = 0$, $Z_0\Omega^q_X = \Omega^q_X$, $B_1\Omega^q_X = B\Omega^q_X$, $Z_1\Omega^q_X = Z\Omega^q_X$, and where for $s \geqslant 2$, $B_s\Omega^q_X$ and $Z_s\Omega^q_X$ are defined to be the subsheaves of abelian groups of Ω^q_X characterized by the short exact sequences

$$\begin{aligned} 0 \to B_1\Omega^q_X \to B_s\Omega^q_X \xrightarrow{C} B_{s-1}\Omega^q_X \to 0 \\ 0 \to B_1\Omega^q_X \to Z_s\Omega^q_X \xrightarrow{C} Z_{s-1}\Omega^q_X \to 0 \end{aligned} \tag{3.3}$$

It then follows from [16, Corollary I.3.9] that there is a short exact sequence

$$0 \to \Omega^q_X/B_{n-1}\Omega^q_X \to \mathrm{gr}^{n-1} W_n\Omega^q_X \to \Omega^{q-1}_X/Z_{n-1}\Omega^{q-1}_X \to 0, \tag{3.4}$$

of sheaves of abelian groups on Sm/X for the étale topology.

Proposition 3.5. *Let k be a perfect field of positive characteristic p and assume that resolution of singularities holds over k. Then for all schemes X essentially smooth over k and all integers $n \geqslant 1$ and $q \geqslant 0$, the change-of-topology maps*

$$\begin{tikzcd} H^*_{\mathrm{Zar}}(X, a_{\mathrm{Zar}}W_n\Omega^q_X) \arrow[r] \arrow[d] & H^*_{\mathrm{et}}(X, a_{\mathrm{et}}W_n\Omega^q_X) \arrow[d] \\ H^*_{\mathrm{cdh}}(X, a_{\mathrm{cdh}}W_n\Omega^q_X) \arrow[r] & H^*_{\mathrm{eh}}(X, a_{\mathrm{eh}}W_n\Omega^q_X) \end{tikzcd}$$

are isomorphisms.

Proof. Since $a_{\mathrm{Zar}}W_n\Omega^q_X$ and $a_{\mathrm{et}}W_n\Omega^q_X$ are quasi-coherent $W_n(\mathscr{O}_X)$-modules, the top horizontal map is an isomorphism. We show that the right-hand vertical map is an isomorphism; the proof for the left-hand vertical map is analogous. By Lemma 3.1, it suffices to show that the change-of-topology map

$$H^*_{\mathrm{et}}(X, a_{\mathrm{et}}\phi_p W_n\Omega^q_X) \to H^*_{\mathrm{eh}}(X, a_{\mathrm{eh}}\phi_p W_n\Omega^q_X)$$

is an isomorphism. We again abuse notation and write $W_n\Omega^q_X$ for the étale sheaf $a_{\mathrm{et}}\phi_p W_n\Omega^q_X$ on Sm/X.

We proceed by induction on $n \geqslant 1$ as in [4, Proposition 6.3]. In the case $n = 1$, we let $R_{\mathrm{et}}\Gamma(-,\Omega^q_X)$ be the presheaf of chain complexes given by a functorial model for the total right derived functor for the étale topology of the functor $\Gamma(-,\Omega^q_X)$ from Sm/X to the category of abelian groups. We must show that the presheaf $R_{\mathrm{et}}\Gamma(-,\Omega^q_X)$ satisfies descent for the eh-topology. To prove this, it suffices by [4, Corollary 3.9] to show that for every blow-up square of smooth X-schemes

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\ Z \arrow[r] & Y, \end{tikzcd}$$

the induced square of complexes of abelian groups

$$\begin{tikzcd} R_{\mathrm{et}}\Gamma(Z',\Omega^q_X) & R_{\mathrm{et}}\Gamma(Y',\Omega^q_X) \arrow[l] \\ R_{\mathrm{et}}\Gamma(Z,\Omega^q_X) \arrow[u] & R_{\mathrm{et}}\Gamma(Y,\Omega^q_X) \arrow[l] \arrow[u] \end{tikzcd}$$

is homotopy cartesian. But this is proved in [9, Chapter IV, Theorem 1.2.1].

We next assume the statement for $n-1$ and prove it for n. By a five-lemma argument based on the short exact sequence of sheaves (3.2), it suffices to show that the change-of-topology map

$$H^*_{\mathrm{et}}(X,\mathrm{gr}^{n-1} W_n\Omega^q_X) \to H^*_{\mathrm{eh}}(X,\alpha^*\,\mathrm{gr}^{n-1} W_n\Omega^q_X)$$

is an isomorphism. Furthermore, by a five-lemma argument based on the exact sequence (3.4), it suffices to show that for all $s \geqslant 0$, the change-of-topology maps

$$\begin{aligned} H^*_{\mathrm{et}}(X,B_s\Omega^q_X) &\to H^*_{\mathrm{eh}}(X,\alpha^*B_s\Omega^q_X) \\ H^*_{\mathrm{et}}(X,Z_s\Omega^q_X) &\to H^*_{\mathrm{eh}}(X,\alpha^*Z_s\Omega^q_X) \end{aligned} \tag{3.6}$$

are isomorphisms. The case $s = 0$ was proved above. To prove the case $s = 1$, we argue by induction on $q \geqslant 0$: The basic case $q = 0$ has already been proved, since $B_1\Omega^0_X = 0$ and $C\colon Z_1\Omega^0_X = Z_1\Omega^0_X/B_1\Omega^0_X \xrightarrow{\sim} \Omega^0_X$, and the induction step follows by a five-lemma argument based on the exact sequences of sheaves

$$0 \to Z_1\Omega^{q-1}_X \to \Omega^{q-1}_X \xrightarrow{d} B_1\Omega^q_X \to 0$$
$$0 \to B_1\Omega^q_X \to Z_1\Omega^q_X \xrightarrow{C} \Omega^q_X \to 0.$$

Finally, we let $s \geqslant 2$ and assume, inductively, that the maps (3.6) have been proved to be isomorphisms for $s-1$. Then a five-lemma argument based on the short exact sequence of sheaves (3.3) show that the maps (3.6) are isomorphisms for s. This completes the proof. □

Lemma 3.7. *Let X be a noetherian $\mathbb{F}_p$-scheme and let $x \in X$ be a point. Then the canonical map $f\colon \operatorname{Spec}\mathscr{O}_{X,x} \to X$ induces an isomorphism*

$$f^* a_{\mathrm{Zar}} W_n\Omega^q_X \to a_{\mathrm{Zar}} W_n\Omega^q_{\operatorname{Spec}\mathscr{O}_{X,x}}$$

of sheaves of abelian groups on $\mathrm{Sch}/\operatorname{Spec}\mathscr{O}_{X,x}$ for the Zariski topology.

Proof. Let $\mathscr{X}$ be a topos. We recall from [16, Definition I.1.4] that, by definition, the de Rham-Witt complex is the left adjoint of the forgetful functor g from the category of V-pro-complexes in $\mathscr{X}$ to the category of $\mathbb{F}_p$-algebras in $\mathscr{X}$. We let $x \in U \subset X$ be an open neighborhood, and let i_U be the corresponding point of the topos $(\mathrm{Sch}/X)^\wedge$ given by $i_U^*(F) = \Gamma(U,F)$ and $i_{U*}(E)(X') = E^{\mathrm{Hom}_X(U,X')}$. Since $g \circ i_{U*} = i_{U*} \circ g$, we conclude that there is a natural isomorphism

$$\Gamma(U, W_n\Omega^q_X) = \Gamma(U, W_n\Omega^q_{\mathscr{O}_X}) \xrightarrow{\sim} W_n\Omega^q_{\Gamma(U,\mathscr{O}_X)}.$$

Now, let $Y = \operatorname{Spec}\mathscr{O}_{X,x}$. To prove the lemma, it suffices to show that for every scheme Y' affine and of finite type over Y, the canonical map

$$\Gamma(Y', f^p W_n\Omega^q_X) \to \Gamma(Y', W_n\Omega^q_Y)$$

is an isomorphism. The discussion preceeding Proposition 2.2 shows that there exists an affine open neighborhood $x \in U \subset X$ and a scheme U' affine and of finite type over U with $Y' = Y \times_U U'$ such that the map in question is the canonical map

$$\operatorname*{colim}_{x \in V \subset U} \Gamma(V \times_U U', W_n\Omega^q_X) \to \Gamma(Y \times_U U', W_n\Omega^q_Y).$$

Here the colimit ranges over the affine open neighborhoods $x \in V \subset U$. Now, by what was proved above, we may identify this map with the canonical map

$$\operatorname*{colim}_{x \in V \subset U} W_n\Omega^q_{\Gamma(V \times_U U', \mathscr{O}_X)} \to W_n\Omega^q_{\Gamma(Y \times_U U', \mathscr{O}_Y)}.$$

Here the left-hand side is the colimit in the category of sets. However, since the index category for the colimit is filtered, the left-hand side is also equal to the colimit in the category of V-pro-complexes in the category of sets. Therefore, the canonical map in question is an isomorphism. Indeed, being a left adjoint, the de Rham-Witt complex preserves colimits, and the canonical map of $\mathbb{F}_p$-algebras

$$\operatorname*{colim}_{x \in V \subset U} \Gamma(V \times_U U', \mathscr{O}_X) \to \Gamma(Y \times_U U', \mathscr{O}_Y)$$

is an isomorphism. ∎

Theorem 3.8. *Let k be a perfect field of characteristic $p > 0$ such that resolution of singularities holds over k, and let X be a d-dimensional scheme essentially of finite type over k. Then the change-of-topology map*

$$H^i_{\mathrm{et}}(X, a_{\mathrm{et}} W_n\Omega^q_X) \to H^i_{\mathrm{eh}}(X, a_{\mathrm{eh}} W_n\Omega^q_X)$$

is a surjection if $i = d$, and both groups vanish if $i > d$.

Proof. We follow the proof of [4, Theorem 6.1]. Since the sheaves $a_{\mathrm{Zar}}W_n\Omega^q_X$ and $a_{\mathrm{et}}W_n\Omega^q_X$ are quasi-coherent $W_n(\mathscr{O}_X)$-modules, the change-of-topology map defines an isomorphism

$$H^i_{\mathrm{Zar}}(X, a_{\mathrm{Zar}}W_n\Omega^q_X) \xrightarrow{\sim} H^i_{\mathrm{et}}(X, a_{\mathrm{et}}W_n\Omega^q_X).$$

In particular, the common group vanishes for $i > d$. We consider the following commutative diagram, where the horizontal maps are the change-of-topology maps, and where the vertical maps are induced from the canonical closed immersion.

$$\begin{tikzcd}
H^i_{\mathrm{Zar}}(X, a_{\mathrm{Zar}}W_n\Omega^q_X) \arrow[r] \arrow[d] & H^i_{\mathrm{eh}}(X, a_{\mathrm{eh}}W_n\Omega^q_X) \arrow[d] \\
H^i_{\mathrm{Zar}}(X^{\mathrm{red}}, a_{\mathrm{Zar}}W_n\Omega^q_{X^{\mathrm{red}}}) \arrow[r] & H^i_{\mathrm{eh}}(X^{\mathrm{red}}, a_{\mathrm{eh}}W_n\Omega^q_{X^{\mathrm{red}}}).
\end{tikzcd}$$

By Lemma 2.1, the right-hand vertical map is an isomorphism for all $i \geqslant 0$. Moreover, the cohomology groups on the left-hand side may be calculated on the small Zariski sites and X and X^{red}, respectively, and the left-hand vertical map may be identified with the map of sheaf cohomology groups of the small Zariski site of X induced by the surjective map of sheaves

$$a_{\mathrm{Zar}}W_n\Omega^q_X \to i_* a_{\mathrm{Zar}}W_n\Omega^q_{X^{\mathrm{red}}}$$

induced by the closed immersion $i\colon X^{\mathrm{red}} \to X$. It follows that the left-hand vertical map is a surjection for $i \geqslant d$. Therefore, we may assume that X is reduced.

We proceed by induction on d. The case $d = 0$ follows from Proposition 3.5, since every reduced 0-dimensional scheme X of finite type over the perfect field k is smooth over k. So we let $d > 0$ and assume that the statement for schemes of smaller dimension. We must show that the statement holds for every reduced d-dimensional scheme X essentially of finite type over k. Suppose first that X is affine, and hence, separated. By resolution of singularities, there exists an abstract blow-up square

$$\begin{tikzcd}
Y' \arrow[r, "i'"] \arrow[d, "p'"] & X' \arrow[d, "p"] \\
Y \arrow[r, "i"] & X
\end{tikzcd}$$

where X' is essentially smooth over k and where the dimensions of Y and Y' are smaller than d. The group $H^i_{\mathrm{eh}}(X', a_{\mathrm{eh}}W_n\Omega^q_{X'})$ vanishes for $i > d$ by Proposition 3.5 and the groups $H^i_{\mathrm{eh}}(Y, a_{\mathrm{eh}}W_n\Omega^q_Y)$ and $H^i_{\mathrm{eh}}(Y', a_{\mathrm{eh}}W_n\Omega^q_{Y'})$ vanish for $i > d-1$ by the induction. Therefore, the Mayer-Vietoris long exact sequence of eh-cohomology groups associated with the abstract blow-up square above

$$\cdots \to H^{i-1}_{\mathrm{eh}}(Y', a_{\mathrm{eh}}W_n\Omega^q_{Y'}) \to H^i_{\mathrm{eh}}(X, a_{\mathrm{eh}}W_n\Omega^q_X) \to \begin{array}{c} H^i_{\mathrm{eh}}(X', a_{\mathrm{eh}}W_n\Omega^q_{X'}) \\ \oplus \\ H^i_{\mathrm{eh}}(Y, a_{\mathrm{eh}}W_n\Omega^q_Y) \end{array} \to \cdots$$

shows that the group $H^i_{\mathrm{eh}}(X, a_{\mathrm{eh}} W_n \Omega^q_X)$ vanishes for $i > d$ as stated. We must also show that $H^d_{\mathrm{eh}}(X, a_{\mathrm{eh}} W_n \Omega^q_X)$ is zero. We first show that the common group

$$H^d_{\mathrm{Zar}}(X', a_{\mathrm{Zar}} W_n \Omega^q_{X'}) \xrightarrow{\sim} H^d_{\mathrm{eh}}(X', a_{\mathrm{eh}} W_n \Omega^q_{X'})$$

is zero. The left-hand group may be evaluated on the small Zariski site of X'. Now, the theorem of formal functions [10, Corollary III.4.2.2] shows that for every quasi-coherent $W_n(\mathscr{O}'_X)$-module F on the small Zariski site of X', the higher direct image sheaf $R^d p_* F$ on the small Zariski site of X is zero. Since X is affine, we conclude from the Leray spectral sequence that the $H^d_{\mathrm{Zar}}(X', a_{\mathrm{Zar}} W_n \Omega^q_{X'})$ is zero as desired. We next show that the lower horizontal map in the following diagram is surjective.

$$\begin{array}{ccc}
H^{d-1}_{\mathrm{Zar}}(X', a_{\mathrm{Zar}} W_n \Omega^q_{X'}) & \xrightarrow{i'^*} & H^{d-1}_{\mathrm{Zar}}(Y', a_{\mathrm{Zar}} W_n \Omega^q_{Y'}) \\
\downarrow & & \downarrow \\
H^{d-1}_{\mathrm{eh}}(X', a_{\mathrm{eh}} W_n \Omega^q_{X'}) & \xrightarrow{i'^*} & H^{d-1}_{\mathrm{eh}}(Y', a_{\mathrm{eh}} W_n \Omega^q_{Y'})
\end{array}$$

Here the vertical maps are the change-of-topology maps. By induction, the right-hand vertical map is surjective, so we may instead show that the upper horizontal map is surjective. The cohomology groups in the upper row may be evaluated on small Zariski sites of X' and Y', respectively. Moreover, the theorem of formal functions shows that the $R^{d-1} p_*$ is a right-exact functor from the category of quasi-coherent $W_n(\mathscr{O}_{X'})$-modules to the category of quasi-coherent $W_n(\mathscr{O}_X)$-modules. Since the closed immersion i' gives rise to a surjection

$$W_n \Omega^q_{X'} \to i'_* W_n \Omega^q_{Y'}$$

of quasi-coherent $W_n(\mathscr{O}_{X'})$-modules on the small Zariski site of X', we conclude that the induced map

$$R^{d-1} p_* W_n \Omega^q_{X'} \to R^{d-1} p_* i'_* W_n \Omega^q_{Y'}$$

is a surjection of $W_n(\mathscr{O}_X)$-modules on the small Zariski site of X. As X is assumed to be affine, the Leray spectral sequence shows that

$$i'^* \colon H^{d-1}_{\mathrm{Zar}}(X', a_{\mathrm{Zar}} W_n \Omega^q_{X'}) \to H^{d-1}_{\mathrm{Zar}}(Y', a_{\mathrm{Zar}} W_n \Omega^q_{Y'})$$

is surjective as desired. We conclude from the Mayer-Vietoris exact sequence that the group $H^d_{\mathrm{eh}}(X, a_{\mathrm{eh}} W_n \Omega^q_X)$ is zero. This proves the statement of the theorem for X a d-dimensional reduced affine scheme essentially of finite type over k.

It remains to prove the statement for X a general d-dimensional reduced scheme essentially of finite type over k. To this end, we let $\varepsilon \colon (\mathrm{Sch}/X)_{\mathrm{eh}} \to (\mathrm{Sch}/X)_{\mathrm{Zar}}$ be the canonical morphism of sites and consider the Leray spectral sequence

$$E_2^{s,t} = H^s_{\mathrm{Zar}}(X, R^t \varepsilon_* a_{\mathrm{eh}} W_n \Omega^q_X) \Rightarrow H^{s+t}_{\mathrm{eh}}(X, a_{\mathrm{eh}} W_n \Omega^q_X).$$

Let $x \in X$ be a point of codimension c. Then Proposition 2.2 and Lemma 3.7 show that the stalk of $R^t \varepsilon_* a_{\mathrm{eh}} W_n \Omega^q_X$ at x is given by

$$(R^t \varepsilon_* a_{\mathrm{eh}} W_n \Omega^q_X)_x = H^t_{\mathrm{eh}}(\operatorname{Spec} \mathscr{O}_{X,x}, a_{\mathrm{eh}} W_n \Omega^q_{\operatorname{Spec} \mathscr{O}_{X,x}}).$$

We have proved that this group vanishes if either $c > 0$ and $t \geqslant c$ or $c = 0$ and $t > 0$, or equivalently, if $t > 0$ and $c \leqslant t$. It follows that for $t > 0$, the higher direct image sheaf $R^t\varepsilon_* a_{\mathrm{eh}} W_n\Omega_X^q$ is supported in dimension $< d-t$. Hence, $E_2^{s,t}$ vanishes if $t > 0$ and $s+t \geqslant d$. This shows that $H^i_{\mathrm{eh}}(X, a_{\mathrm{eh}} W_n\Omega_X^q)$ is zero for $i > d$ and that the edge homomorphism defines a surjection

$$H^d_{\mathrm{Zar}}(X, \varepsilon_* a_{\mathrm{eh}} W_n\Omega_X^q) \twoheadrightarrow H^d_{\mathrm{eh}}(X, a_{\mathrm{eh}} W_n\Omega_X^q).$$

Finally, it follows from Proposition 3.5 that the cokernel of the unit map

$$a_{\mathrm{Zar}} W_n\Omega_X^q \to \varepsilon_* a_{\mathrm{eh}} W_n\Omega_X^q$$

is supported on the singular set of X which has dimension strictly less that d. Since the functor $H^d_{\mathrm{Zar}}(X,-)$ is right-exact, we conclude that the induced map

$$H^d_{\mathrm{Zar}}(X, a_{\mathrm{Zar}} W_n\Omega_X^q) \to H^d_{\mathrm{Zar}}(X, \varepsilon_* a_{\mathrm{eh}} W_n\Omega_X^q)$$

is surjective. This proves the induction step and the theorem. $\square$

4 The sheaves $a_{\mathrm{eh}}\,\mathrm{TR}^n_q(-;p)$ and $a_{\mathrm{eh}}\,\mathrm{TC}^n_q(-;p)$

Let X be a noetherian $\mathbb{F}_p$-scheme. We briefly recall the presheaves of fibrant symmetric spectra $\mathrm{TR}^n(-;p)$ and $\mathrm{TC}^n(-;p)$ on Sch/X and refer to [8] and [2] for a detailed discussion. Topological Hochschild homology gives a presheaf $\mathrm{THH}(-)$ of fibrant symmetric spectra with an action by the multiplicative group $\mathbb{T}$ of complex numbers of modulus 1. We let $C_{p^{n-1}} \subset \mathbb{T}$ be the subgroup of order p^{n-1} and define

$$\mathrm{TR}^n(-;p) = \mathrm{THH}(-)^{C_{p^{n-1}}}$$

to be the presheaf of fibrant symmetric spectra given by the $C_{p^{n-1}}$-fixed points. There are two maps of presheaves of fibrant symmetric spectra

$$R, F\colon\ \mathrm{TR}^n(-;p) \to \mathrm{TR}^{n-1}(-;p)$$

called the restriction map and the Frobenius map, respectively, and the presheaf of fibrant symmetric spectra $\mathrm{TC}^n(-;p)$ is defined to be their homotopy equalizer. It follows immediately from the definition that the associated presheaves of homotopy groups are related by a long exact sequence

$$\cdots \longrightarrow \mathrm{TC}^n_q(X;p) \longrightarrow \mathrm{TR}^n_q(X;p) \xrightarrow{R-F} \mathrm{TR}^{n-1}_q(X;p) \longrightarrow \cdots$$

Moreover, by [14, Proposition 6.2.4], the sheaves $a_{\mathrm{Zar}}\,\mathrm{TR}^n_q(-;p)$ and $a_{\mathrm{et}}\,\mathrm{TR}^n_q(-;p)$ agree and are quasi-coherent $W_n(\mathscr{O}_X)$-modules on Sch/X.

Proposition 4.1. *Let k be a perfect field of positive characteristic p such that resolution of singularities holds over k, and let X be a scheme essentially smooth over k. Then for all integers q and $n \geqslant 1$, the change-of-topology maps*

$$\begin{tikzcd}
H^*_{\mathrm{Zar}}(X, a_{\mathrm{Zar}}\,\mathrm{TR}^n_q(-;p)) \arrow[r] \arrow[d] & H^*_{\mathrm{et}}(X, a_{\mathrm{et}}\,\mathrm{TR}^n_q(-;p)) \arrow[d] \\
H^*_{\mathrm{cdh}}(X, a_{\mathrm{cdh}}\,\mathrm{TR}^n_q(-;p)) \arrow[r] & H^*_{\mathrm{eh}}(X, a_{\mathrm{eh}}\,\mathrm{TR}^n_q(-;p))
\end{tikzcd}$$

are isomorphisms.

Proof. The sheaves $a_{\mathrm{Zar}}\,\mathrm{TR}^n_q(-;p)$ and $a_{\mathrm{et}}\,\mathrm{TR}^n_q(-;p)$ are sheaves of quasi-coherent $W_n(\mathscr{O}_X)$-modules. Therefore, the top horizontal map is an isomorphism. We show that the right-hand vertical map is an isomorphism; the proof for the left-hand vertical map is analogous. It suffices by Lemma 3.1, to show that the change-of-topology map

$$H^*_{\mathrm{et}}(X, a_{\mathrm{et}}\phi_p\,\mathrm{TR}^n_q(-;p)) \to H^*_{\mathrm{eh}}(X, a_{\mathrm{eh}}\phi_p\,\mathrm{TR}^n_q(-;p))$$

is an isomorphism. We recall from [12, Theorem B] that there is a canonical isomorphism of sheaves of abelian groups on Sm/X for the Zariski topology

$$\bigoplus_{m \geqslant 0} a_{\mathrm{Zar}}\phi_p W_n\Omega^{q-2m}_X \xrightarrow{\sim} a_{\mathrm{Zar}}\phi_p\,\mathrm{TR}^n_q(-;p).$$

It induces an isomorphism of associated sheaves for the étale topology and for the eh-topology. The proposition now follows from Proposition 3.5 above. $\square$

Corollary 4.2. *Let k be a perfect field of positive characteristic p and assume that resolution of singularities holds over k. Let X be a d-dimensional scheme essentially of finite type over k. Then for all integers q and $n \geqslant 1$,*

$$H^{d+1}_{\mathrm{eh}}(X, a_{\mathrm{eh}}\,\mathrm{TR}^n_q(-;p)) = 0.$$

Proof. The proof is by induction on d and is analogous to the first part of the proof of Theorem 3.8 above. $\square$

Let k be a perfect field of characteristic $p > 0$ and let X be a scheme essentially of finite type over k. We consider the long exact sequence

$$\cdots \longrightarrow \{a_{\mathrm{et}}\,\mathrm{TC}^n_q(-;p)\} \longrightarrow \{a_{\mathrm{et}}\,\mathrm{TR}^n_q(-;p)\} \xrightarrow{\mathrm{id}-F} \{a_{\mathrm{et}}\,\mathrm{TR}^n_q(-;p)\} \longrightarrow \cdots$$

of étale sheaves of pro-abelian groups on Sch/X. Here the structure maps in the pro-abelian groups are the restriction maps R. We recall from [13] that there is a canonical map compatible with restriction and Frobenius operators

$$a_{\mathrm{et}}W_n\Omega^q_X \to a_{\mathrm{et}}\,\mathrm{TR}^n_q(-;p) \tag{4.3}$$

and that this map is an isomorphism, for $q \leqslant 1$. We remark that the assumption in op. cit. that p be odd is unnecessary. Indeed, [5, Theorem 4.3] shows that the result is valid also for $p = 2$. We examine the map $\mathrm{id} - F$ in degrees $q \leqslant 1$.

Lemma 4.4. *Let X be a noetherian scheme over $\mathbb{F}_p$. Then there is an exact sequence of sheaves of pro-abelian groups on* Sch/X *for the étale topology:*

$$0 \longrightarrow \{\mathbb{Z}/p^n\mathbb{Z}\} \longrightarrow \{a_{\mathrm{et}}W_n(\mathscr{O}_X)\} \xrightarrow{\mathrm{id}-F} \{a_{\mathrm{et}}W_n(\mathscr{O}_X)\} \longrightarrow 0.$$

Proof. Since X is a scheme over $\mathbb{F}_p$, the Frobenius map F agrees with the map $\{W_n(\varphi)\}$ induced by the Frobenius endomorphism of X. We prove that for every strictly henselian noetherian local $\mathbb{F}_p$-algebra $(A,\mathfrak{m},\kappa)$, the sequence

$$0 \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow W_n(A) \xrightarrow{\mathrm{id}-W_n(\varphi)} W_n(A) \longrightarrow 0$$

is exact. Since A is strictly henselian, the map $\mathrm{id}-\varphi\colon A \to A$ is surjective. An induction argument based on the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{V^{n-1}} & W_n(A) & \xrightarrow{R} & W_{n-1}(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}-\varphi} & & \downarrow{\scriptstyle \mathrm{id}-W_n(\varphi)} & & \downarrow{\scriptstyle \mathrm{id}-W_{n-1}(\varphi)} & & \\
0 & \longrightarrow & A & \xrightarrow{V^{n-1}} & W_n(A) & \xrightarrow{R} & W_{n-1}(A) & \longrightarrow & 0
\end{array}$$

shows that the map $\mathrm{id}-F$ is surjective as stated. To identify the kernel of $\mathrm{id}-F$, we consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_n(\mathfrak{m}) & \longrightarrow & W_n(A) & \longrightarrow & W_n(\kappa) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}-W_n(\varphi)} & & \downarrow{\scriptstyle \mathrm{id}-W_n(\varphi)} & & \downarrow{\scriptstyle \mathrm{id}-W_n(\varphi)} & & \\
0 & \longrightarrow & W_n(\mathfrak{m}) & \longrightarrow & W_n(A) & \longrightarrow & W_n(\kappa) & \longrightarrow & 0
\end{array}$$

We wish to show that the unit map $\eta\colon W_n(\mathbb{F}_p) \to W_n(A)$ is an isomorphism onto the kernel of the middle vertical map. We have
$operatorname im(\eta) \subset \ker(\mathrm{id}-W_n(\varphi))$. Moreover, since κ is a domain, the composition

$$W_n(\mathbb{F}_p) \to W_n(A) \to W_n(\kappa)$$

of η and the canonical projection is an isomorphism of $W_n(\mathbb{F}_p)$ onto the kernel of the right-hand vertical map in the diagram above. Hence, it will suffice to show that, in the diagram above, the left-hand vertical map is injective. Moreover, an induction argument based on the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_{n-1}(\mathfrak{m}) & \xrightarrow{V} & W_n(\mathfrak{m}) & \xrightarrow{R^{n-1}} & \mathfrak{m} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}-W_{n-1}(\varphi)} & & \downarrow{\scriptstyle \mathrm{id}-W_n(\varphi)} & & \downarrow{\scriptstyle \mathrm{id}-\varphi} & & \\
0 & \longrightarrow & W_{n-1}(\mathfrak{m}) & \xrightarrow{V} & W_n(\mathfrak{m}) & \xrightarrow{R^{n-1}} & \mathfrak{m} & \longrightarrow & 0
\end{array}$$

shows that it suffices to consider the case $n=1$. In this case, we recall that by a theorem of Krull [10, Corollary 0.7.3.6], the $\mathfrak{m}$-adic topology on A is separated. It follows that the map $\mathrm{id}-\varphi\colon \mathfrak{m} \to \mathfrak{m}$ is injective as desired. $\square$

Lemma 4.5. *Let X be a noetherian scheme over $\mathbb{F}_p$. Then the map of sheaves of pro-abelian groups on* Sch/X *for the étale topology*

$$\mathrm{id} - F\colon \{a_{\mathrm{et}} W_n \Omega^1_X\} \to \{a_{\mathrm{et}} W_n \Omega^1_X\}$$

is an epimorphism.

Proof. It will suffice to show that for every strictly henselian noetherian local $\mathbb{F}_p$-algebra $(A, \mathfrak{m}, \kappa)$, the map

$$R - F\colon W_n \Omega^1_A \to W_{n-1} \Omega^1_A$$

is surjective. Since A is local, the abelian group $W_n \Omega^1_A$ is generated by elements of the form $ad\log[x]_n$ with $a \in W_n(A)$ and $x \in 1 + \mathfrak{m}$. Moreover, we have

$$(R-F)(ad\log[x]_n) = (R-F)(a)d\log[x]_{n-1}.$$

Hence, the proof of Lemma 4.4 shows that $R - F$ is an epimorphism. The lemma follows. □

Proposition 4.6. *Let X be a noetherian scheme over $\mathbb{F}_p$. Then:*

(i) The sheaf of pro-abelian groups $\{a_{\mathrm{et}}\,\mathrm{TC}^n_q(-;p)\}$ is zero for $q < 0$.
(ii) The sheaf of pro-abelian groups $\{a_{\mathrm{et}}\,\mathrm{TC}^n_0(-;p)\}$ is canonically isomorphic to the sheaf of pro-abelian groups $\{\mathbb{Z}/p^n\mathbb{Z}\}$.
(iii) There is a long exact sequence of sheaves of pro-abelian groups

$$\cdots \longrightarrow \{a_{\mathrm{et}}\,\mathrm{TC}^n_1(-;p)\} \longrightarrow \{a_{\mathrm{et}}\,\mathrm{TR}^n_1(-;p)\} \xrightarrow{\mathrm{id}-F} \{a_{\mathrm{et}}\,\mathrm{TR}^n_1(-;p)\} \longrightarrow 0$$

Proof. This follows immediately from the fact that the map (4.3) is an isomorphism, for $q \leqslant 1$, and from Lemmas 4.4 and 4.5. □

Question 4.7. We do not know whether or not the sequence of sheaves

$$0 \to \{a_{\mathrm{et}}\,\mathrm{TC}^n_q(-;p)\} \to \{a_{\mathrm{et}}\,\mathrm{TR}^n_q(-;p)\} \xrightarrow{1-F} \{a_{\mathrm{et}}\,\mathrm{TR}^n_q(-;p)\} \to 0$$

on Sch/X for the étale topology is exact for $q \geqslant 1$.

Theorem 4.8. *Let k be a field of positive characteristic p and assume that resolution of singularities holds over k. Let X be a d-dimensional scheme essentially of finite type over k. Then the map of pro-abelian groups*

$$\{H^i_{\mathrm{et}}(X, a_{\mathrm{et}}\,\mathrm{TC}^n_q(-;p))\} \to \{H^i_{\mathrm{eh}}(X, a_{\mathrm{eh}}\,\mathrm{TC}^n_q(-;p))\}$$

induced by the change-of-topology maps is an epimorphism if $q = 1$ and $i = d+1$, an isomorphism if $q = 0$, and the two pro-abelian groups are zero if $q \leqslant -1$.

Proof. The statement for $q < 0$ follows from Proposition 4.6 (i), and the statement for $q = 0$ follows from Proposition 4.6 (ii) and from [6, Theorem 3.6] which shows that for the constant sheaf $\mathbb{Z}/p^n\mathbb{Z}$, the change-of-topology map

$$H^*_{\mathrm{et}}(X,\mathbb{Z}/p^n\mathbb{Z}) \to H^*_{\mathrm{eh}}(X,\mathbb{Z}/p^n\mathbb{Z})$$

is an isomorphism. To prove the statement for $q = 1$, we fix a d-dimensional scheme X essentially of finite type over k. The p-cohomological dimension of X with respect to both the étale topology and the eh-topology is less than or equal to $d+1$; see Theorem D. Let us write

$$\cdots \to \{F^n_{-2}\} \to \{F^n_{-1}\} \to \{F^n_0\} \to 0$$

for the long exact sequence of étale sheaves of pro-abelian groups on Sch/X from Proposition 4.6 (iii). Then we have hypercohomology spectral sequences

$$E_1^{s,t} = \{H^t_{\mathrm{et}}(X,F^n_{-s})\} \Rightarrow \{\mathbb{H}^{s+t}_{\mathrm{et}}(X,F^n_\cdot)\}$$
$$E_1^{s,t} = \{H^t_{\mathrm{eh}}(X,\alpha^*F^n_{-s})\} \Rightarrow \{\mathbb{H}^{s+t}_{\mathrm{eh}}(X,\alpha^*F^n_\cdot)\}$$

with the d_1-differentials induced by the differential in the cochain complex $\{F^n_\cdot\}$. Since the complex $\{F^n_\cdot\}$ is exact and since the cohomological dimension of X is bounded, the spectral sequences converge and their abutment is zero. Moreover, the change-of-topology maps induce a map of spectral sequences from the top spectral sequence to the bottom spectral sequence. We proved in Corollary 4.2 that the cohomology groups $H^{d+1}_{\mathrm{eh}}(X,\alpha^*F^n_{-3})$ and $H^{d+1}_{\mathrm{eh}}(X,\alpha^*F^n_{-1})$ vanish. Similarly, the cohomology groups $H^{d+1}_{\mathrm{et}}(X,F^n_{-3})$ and $H^{d+1}_{\mathrm{et}}(X,F^n_{-1})$ vanish, since the sheaves F^n_{-3} and F^n_{-1} are quasi-coherent $W_n(\mathscr{O}_X)$-modules. Hence, the map of hypercohomology spectral sequences gives rise to a commutative diagram

$$\begin{tikzcd}
\{H^d_{\mathrm{et}}(X,F^n_{-1})\} \arrow[r, "\mathrm{id}-F"] \arrow[d] & \{H^d_{\mathrm{et}}(X,F^n_0)\} \arrow[r] \arrow[d] & \{H^{d+1}_{\mathrm{et}}(X,F^n_{-2})\} \arrow[r] \arrow[d] & 0 \\
\{H^d_{\mathrm{eh}}(X,\alpha^*F^n_{-1})\} \arrow[r, "\mathrm{id}-F"] & \{H^d_{\mathrm{eh}}(X,\alpha^*F^n_0)\} \arrow[r] & \{H^{d+1}_{\mathrm{eh}}(X,\alpha^*F^n_{-2})\} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Here the middle and left-hand vertical maps are both equal to the change-of-topology map

$$\{H^d_{\mathrm{et}}(X,a_{\mathrm{et}}\,\mathrm{TR}^n_1(-;p))\} \to \{H^d_{\mathrm{eh}}(X,a_{\mathrm{eh}}\,\mathrm{TR}^n_1(-;p))\}$$

which by [13] coincides with the change-of-topology map

$$\{H^d_{\mathrm{et}}(X,a_{\mathrm{et}}W_n\Omega^1_X)\} \to \{H^d_{\mathrm{eh}}(X,a_{\mathrm{eh}}W_n\Omega^1_X)\}.$$

We proved in Theorem 3.8 that this map is an epimorphism. Therefore, also the right-hand vertical map,

$$\{H^{d+1}_{\mathrm{et}}(X,a_{\mathrm{et}}\,\mathrm{TC}^n_1(-;p))\} \to \{H^{d+1}_{\mathrm{eh}}(X,a_{\mathrm{eh}}\,\mathrm{TC}^n_1(-;p))\},$$

is an epimorphism. This completes the proof. ∎

5 Proof of Theorems A and C

We first show that, in the statement of Theorem C, we may replace the Zariski topology and cdh-topology by the étale topology and eh-topology, respectively.

Theorem 5.1. *Let k be a perfect field of positive characteristic p and assume that resolution of singularities holds over k. Then for every scheme X essentially of finite type over k, the horizontal maps in the diagram of change-of-topology maps*

$$\begin{array}{ccc}
\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X,\mathrm{TC}^n(-;p)) & \longrightarrow & \mathbb{H}^{\cdot}_{\mathrm{et}}(X,\mathrm{TC}^n(-;p)) \\
\downarrow & & \downarrow \\
\mathbb{H}^{\cdot}_{\mathrm{cdh}}(X,\mathrm{TC}^n(-;p)) & \longrightarrow & \mathbb{H}^{\cdot}_{\mathrm{eh}}(X,\mathrm{TC}^n(-;p)).
\end{array}$$

are weak equivalences.

Proof. The diagram in the statement agrees with the diagram of homotopy equalizers of the maps induced by the restriction and Frobenius maps from the following diagram of change-of-topology maps to itself.

$$\begin{array}{ccc}
\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X,\mathrm{TR}^n(-;p)) & \longrightarrow & \mathbb{H}^{\cdot}_{\mathrm{et}}(X,\mathrm{TR}^n(-;p)) \\
\downarrow & & \downarrow \\
\mathbb{H}^{\cdot}_{\mathrm{cdh}}(X,\mathrm{TR}^n(-;p)) & \longrightarrow & \mathbb{H}^{\cdot}_{\mathrm{eh}}(X,\mathrm{TR}^n(-;p)).
\end{array}$$

Hence, it suffices to prove that the horizontal maps in this diagram are weak equivalences. The top horizontal map induces a map from the spectral sequence

$$E^2_{s,t} = H^{-s}_{\mathrm{Zar}}(X, a_{\mathrm{Zar}}\,\mathrm{TR}^n_t(-;p)) \Rightarrow \mathbb{H}^{-s-t}_{\mathrm{Zar}}(X,\mathrm{TR}^n(-;p))$$

to the spectral sequence

$$E^2_{s,t} = H^{-s}_{\mathrm{et}}(X, a_{\mathrm{et}}\,\mathrm{TR}^n_t(-;p)) \Rightarrow \mathbb{H}^{-s-t}_{\mathrm{et}}(X,\mathrm{TR}^n(-;p)).$$

The map of E^2-terms is given by the change-of-topology map

$$H^{-s}_{\mathrm{Zar}}(X, a_{\mathrm{Zar}}\,\mathrm{TR}^n_t(-;p)) \to H^{-s}_{\mathrm{et}}(X, a_{\mathrm{et}}\,\mathrm{TR}^n_t(-;p))$$

and is an isomorphism, since $a_{\mathrm{Zar}}\,\mathrm{TR}^n_t(-;p)$ and $a_{\mathrm{et}}\,\mathrm{TR}^n_t(-;p)$ are quasi-coherent $W_n(\mathscr{O}_X)$-modules. It follows that the map

$$\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X,\mathrm{TR}^n(-;p)) \to \mathbb{H}^{\cdot}_{\mathrm{et}}(X,\mathrm{TR}^n(-;p))$$

is a weak equivalence as stated. It remains to prove that also the lower horizontal map is a weak equivalence.

Suppose first that X is essentially smooth over k. The left-hand vertical map induces a map from the spectral sequence

$$E^2_{s,t} = H^{-s}_{\mathrm{et}}(X, a_{\mathrm{et}}\,\mathrm{TR}^n_t(-;p)) \Rightarrow \mathbb{H}^{-s-t}_{\mathrm{et}}(X,\mathrm{TR}^n(-;p)).$$

to the spectral sequence

$$E^2_{s,t} = H^{-s}_{\mathrm{eh}}(X, a_{\mathrm{eh}}\,\mathrm{TR}^n_t(-;p)) \Rightarrow \mathbb{H}^{-s-t}_{\mathrm{eh}}(X, \mathrm{TR}^n(-;p)).$$

The map of E^2-terms is the change-of-topology map

$$H^{-s}_{\mathrm{et}}(X, a_{\mathrm{et}}\,\mathrm{TR}^n_t(-;p)) \to H^{-s}_{\mathrm{eh}}(X, a_{\mathrm{eh}}\,\mathrm{TR}^n_t(-;p))$$

which is an isomorphism by Proposition 4.1. Therefore, the change-of-topology map

$$\mathbb{H}^{\cdot}_{\mathrm{et}}(X, \mathrm{TR}^n(-;p)) \to \mathbb{H}^{\cdot}_{\mathrm{eh}}(X, \mathrm{TR}^n(-;p))$$

is a weak equivalence. One shows similarly that the change-of-topology map

$$\mathbb{H}^{\cdot}_{\mathrm{Zar}}(X, \mathrm{TR}^n(-;p)) \to \mathbb{H}^{\cdot}_{\mathrm{cdh}}(X, \mathrm{TR}^n(-;p))$$

is a weak equivalence. Hence, for X essentially smooth over k, the lower horizontal map is a weak equivalence.

Finally, we show that for a general scheme X essentially of finite type over k, the lower horizontal map in the diagram at the beginning of the proof is a weak equivalence. Lemma 2.1 and the appropriate descent spectral sequences show that the vertical maps in the diagram

$$\begin{tikzcd}
\mathbb{H}^{\cdot}_{\mathrm{cdh}}(X, \mathrm{TR}^n(-;p)) \arrow[r] \arrow[d] & \mathbb{H}^{\cdot}_{\mathrm{eh}}(X, \mathrm{TR}^n(-;p)) \arrow[d] \\
\mathbb{H}^{\cdot}_{\mathrm{cdh}}(X^{\mathrm{red}}, \mathrm{TR}^n(-;p)) \arrow[r] & \mathbb{H}^{\cdot}_{\mathrm{eh}}(X^{\mathrm{red}}, \mathrm{TR}^n(-;p))
\end{tikzcd}$$

are weak equivalences. Hence, we may assume that X is reduced. We proceed by induction on the dimension $d = \dim(X)$. The case $d = 0$ has already been proved since every reduced 0-dimensional scheme of finite type over the perfect field k is smooth over k. So let X be a reduced d-dimensional scheme separated and essentially of finite type over k and assume that the statement has been proved for all schemes separated and essentially of finite type over k of dimension at most $d-1$. We may further assume that X is affine, and hence, separated. Therefore, by the assumption of resolution of singularities, there exists an abstract blow-up square

$$\begin{tikzcd}
Y' \arrow[r, "i'"] \arrow[d, "p'"] & X' \arrow[d, "p"] \\
Y \arrow[r, "i"] & X
\end{tikzcd}$$

where is X' essentially smooth over k and where the dimensions of Y and Y' are strictly smaller than d. The change-of-topology map gives rise to a map from the square of symmetric spectra

$$\begin{tikzcd}
\mathbb{H}^{\cdot}_{\mathrm{cdh}}(Y', \mathrm{TR}^n(-;p)) & \mathbb{H}^{\cdot}_{\mathrm{cdh}}(X', \mathrm{TR}^n(-;p)) \arrow[l, "i'^*"'] \\
\mathbb{H}^{\cdot}_{\mathrm{cdh}}(Y, \mathrm{TR}^n(-;p)) \arrow[u, "p'^*"] & \mathbb{H}^{\cdot}_{\mathrm{cdh}}(X, \mathrm{TR}^n(-;p)) \arrow[l, "i^*"'] \arrow[u, "p^*"]
\end{tikzcd}$$

to the square of symmetric spectra

$$\begin{tikzcd}
\mathbb{H}^{\cdot}_{\mathrm{eh}}(Y',\mathrm{TR}^n(-;p)) & \mathbb{H}^{\cdot}_{\mathrm{eh}}(X',\mathrm{TR}^n(-;p)) \arrow[l, "i'^*"'] \\
\mathbb{H}^{\cdot}_{\mathrm{eh}}(Y,\mathrm{TR}^n(-;p)) \arrow[u, "p'^*"'] & \mathbb{H}^{\cdot}_{\mathrm{eh}}(X,\mathrm{TR}^n(-;p)) \arrow[l, "i^*"'] \arrow[u, "p^*"']
\end{tikzcd}$$

both of which are homotopy cartesian. The map of upper right-hand terms is a weak equivalence, since X' is essentially smooth over k. Moreover, since Y and Y' have dimension at most $d-1$, the maps of the two left-hand terms are weak equivalences by the inductive hypothesis. It follows that the map of lower right-hand terms is a weak equivalence. This completes the proof. $\square$

Proof of Theorem C. Let X be a d-dimensional scheme essentially of finite type over the field k. By Theorem 5.1, it suffices to show that the change-of-topology map

$$\{\mathbb{H}^{-q}_{\mathrm{et}}(X,\mathrm{TC}^n(-;p))\} \to \{\mathbb{H}^{-q}_{\mathrm{eh}}(X,\mathrm{TC}^n(-;p))\}$$

is an isomorphism of pro-abelian groups for $q < -d$, and an epimorphism of pro-abelian groups for $q = -d$. We recall from [1, X Theorem 5.1] and Theorem D that the p-cohomological dimension of X for the both the étale topology and the eh-topology is at most $d+1$. In particular, the spectral sequences

$$E^2_{s,t} = H^{-s}_{\mathrm{et}}(X, a_{\mathrm{et}}\,\mathrm{TC}^n_t(-;p)) \Rightarrow \mathbb{H}^{-s-t}_{\mathrm{et}}(X,\mathrm{TC}^n(-;p))$$
$$E^2_{s,t} = H^{-s}_{\mathrm{eh}}(X, a_{\mathrm{et}}\,\mathrm{TC}^n_t(-;p)) \Rightarrow \mathbb{H}^{-s-t}_{\mathrm{eh}}(X,\mathrm{TC}^n(-;p))$$

converge strongly and the induced filtration of the abutment is of finite length less than or equal to $d+1$. Therefore, as n varies, these spectral sequences give rise to strongly convergent spectral sequences of pro-abelian groups

$$E^2_{s,t} = \{H^{-s}_{\mathrm{et}}(X, a_{\mathrm{et}}\,\mathrm{TC}^n_t(-;p))\} \Rightarrow \{\mathbb{H}^{-s-t}_{\mathrm{et}}(X,\mathrm{TC}^n(-;p))\}$$
$$E^2_{s,t} = \{H^{-s}_{\mathrm{eh}}(X, a_{\mathrm{eh}}\,\mathrm{TC}^n_t(-;p))\} \Rightarrow \{\mathbb{H}^{-s-t}_{\mathrm{eh}}(X,\mathrm{TC}^n(-;p))\}.$$

The map in question induces a map of spectral sequences from the top spectral sequence to the bottom spectral sequence which, on E^2-terms, is given by the change-of-topology map in sheaf cohomology. The two E^2-terms are concentrated in the region where $-d-1 \leqslant s \leqslant 0$ and $t \geqslant 0$. Moreover, Theorem 4.8 shows that the change-of-topology map is an isomorphism of pro-abelian groups if $t = 0$, and an epimorphism of pro-abelian groups if $t = 1$ and $s = -d-1$. The theorem follows. $\square$

Proof of Theorem A. Let X be a d-dimensional scheme essentially of finite type over the field k. Then Theorem B shows that the diagram of pro-spectra

$$\begin{tikzcd}
K(X) \arrow[r] \arrow[d] & \mathbb{H}^{\cdot}_{\mathrm{cdh}}(X,K(-)) \arrow[d] \\
\{\mathrm{TC}^n(X;p)\} \arrow[r] & \{\mathbb{H}^{\cdot}_{\mathrm{cdh}}(X,\mathrm{TC}^n(-;p))\}
\end{tikzcd}$$

is homotopy cartesian. We conclude from Theorem C that the canonical map

$$K_q(X) \to \mathbb{H}^{-q}_{\mathrm{cdh}}(X, K(-))$$

is an isomorphism, for $q < -d$, and an epimorphism for $q = -d$. The groups on the right-hand side are the abutment of the spectral sequence

$$E^2_{s,t} = H^s_{\mathrm{cdh}}(X, a_{\mathrm{cdh}}K_t(-)) \Rightarrow \mathbb{H}^{-s-t}_{\mathrm{cdh}}(X, K(-)).$$

The assumption that resolution of singularities holds over k implies that every cdh-covering of an object in Sch/X admits a refinement to a cdh-covering by schemes essentially smooth over k. This, in turn, implies that the sheaf $a_{\mathrm{cdh}}K_t(-)$ vanishes for $t < 0$, and is canonically isomorphic to the constant sheaf $\mathbb{Z}$ for $t = 0$. We recall from [19, Theorem 12.5] that the cdh-cohomological dimension of X is less than or equal to d. Therefore, the groups $E^2_{s,t}$ are zero unless $-d \leqslant s \leqslant 0$ and $t \geqslant 0$. It follows that $\mathbb{H}^{-q}_{\mathrm{cdh}}(X, K(-))$ is zero for $q < -d$, and that $\mathbb{H}^{d}_{\mathrm{cdh}}(X, K(-))$ is canonically isomorphic to $H^d_{\mathrm{cdh}}(X, \mathbb{Z})$. Hence $K_q(X)$ vanishes for $q < -d$ as stated. □

Remark 5.2. The proof above also shows that $K_{-d}(X)$ surjects onto $H^d_{\mathrm{cdh}}(X, \mathbb{Z})$.

Theorem 5.3. *Suppose that strong resolution of singularities holds over every infinite perfect field of positive characteristic p and let X be a d-dimensional scheme of finite type over some field of characteristic p. Then $K_q(X)$ vanishes for $q < -d$.*

Proof. We first let F be a finite field of characteristic p, and let X be a d-dimensional scheme essentially of finite type over F. We let ℓ be a prime number, let F' be a Galois extension of F with Galois group isomorphic to the additive group $\mathbb{Z}_\ell$ of ℓ-adic integers, and let X' be the base-change of X along $\operatorname{Spec} F' \to \operatorname{Spec} F$. Then F' is an infinite perfect field. By assumption, strong resolution of singularities holds over F', so Theorem A shows that $K_q(X')$ vanishes for $q < d$. We claim that the kernel of the pull-back map $K_*(X) \to K_*(X')$ is an ℓ-primary torsion group. To see this, let F'_i be the unique subfield of F' such that $[F'_i : F] = \ell^i$, and let X'_i be the base-change of X along $\operatorname{Spec} F'_i \to \operatorname{Spec} F$. The composition of the pull-back and push-forwards maps

$$K_*(X) \to K_*(X'_i) \to K_*(X)$$

is equal to multiplication by ℓ^i, and hence, the kernel of the pull-back map is annihilated ℓ^i. Moreover, it follows from [10, Section IV.8.5] that the canonical map

$$\operatorname*{colim}_i K_*(X'_i) \to K_*(X')$$

is an isomorphism. Since filtered colimits and finite limits of diagrams of sets commute, we conclude that the kernel of the pull-back map $K_*(X) \to K_*(X')$ is an ℓ-primary torsion group as claimed. It follows that for $q < -d$, $K_q(X)$ is an ℓ-primary torsion group. Since this is true for every prime number ℓ, we find that for $q < -d$, the group $K_q(X)$ is zero.

We let X be a d-dimensional scheme of finite type over an arbitrary field k of characteristic p, and let $k_0 \subset k$ be a perfect subfield. By [10, Theorem IV.8.8.2], there exists an intermediate field $k_0 \subset k_1 \subset k$ such that k_1 is finitely generated over k_0

together with a scheme X_1 of finite type over k_1 such that X is isomorphic to the base-change of X_1 along $\operatorname{Spec} k \to \operatorname{Spec} k_1$. Let $k_1 \subset k_\alpha \subset k$ be a finite generated extension of k_1 contained in k, and let X_α be the base-change of X_1 along $\operatorname{Spec} k_\alpha \to \operatorname{Spec} k_1$. Then X_α is of finite type over k_α. But then X_α is essentially of finite type over k_0. Indeed, the field k_α is the quotient field of a ring A_α of finite type over k_0, and by [10, Theorem IV.8.8.2], we can find a scheme $\mathscr{X}_\alpha$ of finite type over A_α such that X_α is the generic fiber of $\mathscr{X}_\alpha$ over A_α. Therefore, the group $K_q(X_\alpha)$ vanishes for $q < -d$. Finally, it follows from [10, Proposition IV.8.5.5] that the canonical map

$$\operatorname*{colim}_\alpha K_q(X_\alpha) \to K_q(X)$$

from the filtered colimit indexed by all intermediate fields $k_1 \subset k_\alpha \subset k$ finitely generated over k_1 is an isomorphism. Hence, the group $K_q(X)$ vanishes for $q < -d$ as stated. □

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